A Note on Hamilton-Jacobi Formalism and D-brane Effective Actions

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Abstract

We first review the canonical formalism with general space-like hypersurfaces developed by Dirac by rederiving the Hamilton-Jacobi equations which are satisfied by on-shell actions defined on such hypersurfaces. We compare the case of gravitational systems with that of the flat space. Next, we remark as a supplement to our previous results that the effective actions of D-brane and M-brane given by arbitrary embedding functions are on-shell actions of supergravities.

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1 Introduction

We showed in [1, 2] that the effective actions of D-brane and M-brane are on-shell actions of supergravities. We derived the Hamilton-Jacobi (H-J) equations of supergravities, which are satisfied by on-shell actions, regarding a radial direction as time, and solved those equations. We also found that these solutions to the H-J equations are the on-shell actions around the supergravity solutions that are conjectured to be dual to various gauge theories, which in particular include noncommutative super Yang Mills. In the gauge/gravity correspondence, the on-shell actions in gravities are considered to be generating functionals of correlation functions in gauge theories. Therefore, our results in [1, 2] should be useful for going beyond the AdS/CFT correspondence and studying the more general gauge/gravity correspondence.

However, it is not clear why the D-brane effective actions which includes the all-order contributions in the α' expansion are obtained within supergravities, which are just the lowest order approximation for string theories in the α' expansion. In order to clarify this reason, we must establish the exact correspondence between our calculations and the derivations of the D-brane effective action in string theory. On one hand, the worldvolumes of the D-brane effective actions in our solutions are fixed-time hypersurfaces, since the ordinary H-J formalism gives on-shell actions defined on the boundary hypersurfaces specified by the final time. On the other hand, the worldvolumes of D-branes in string theories are defined by embedding functions which can specify arbitrary hypersurfaces in the target space [3]. Therefore, before trying to establish the above correspondence, we should first investigate whether the D-brane effective action whose worldvolume is defined by such embedding functions is an on-shell action of supergravity or not.

Hence, this brief note is concerned with the on-shell action that is obtained by substituting into the action the classical solution which satisfies a boundary condition given on not a fixed-time hypersurface but an general hypersurface. (See Fig.1.) We need a generalization of the H-J formalism that gives such on-shell actions. Indeed, this generalization was studied by Dirac around fifty years ago [4]. The generalization in gravitational systems is much simpler. In fact, by performing a general coordinate transformation that transforms the general hypersurface defined by the embedding functions to a fixed-time hypersurface, one can show that the answer to the above question is positive. However, it is interesting and useful for further developments to compare the case of gravitational system with that of the

flat space, so that we will give a heuristic argument below. We will first derive Dirac's result in the flat space in a different way, and next make the comparison. Finally, we remark that the effective actions of D-brane and M-brane given by arbitrary embedding functions are on-shell actions of supergravities
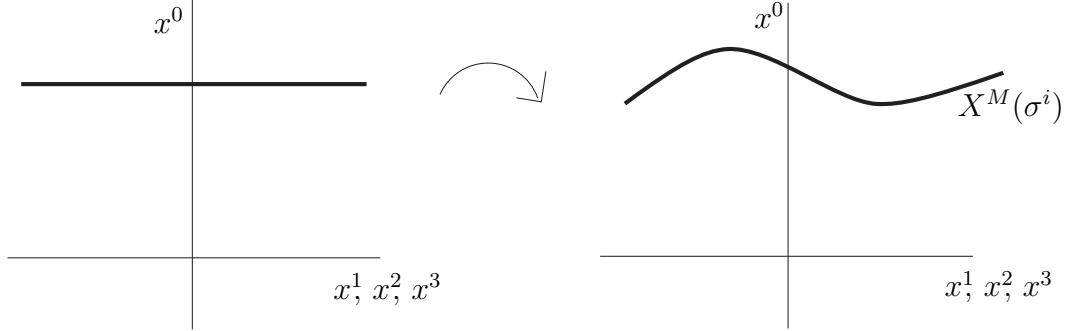

Figure 1: Generalization of a boundary surface

2 Introduction of dynamical coordinates

As we mentioned, we pay attention to on-shell actions with the boundary values of fields given on general space-like hypersurfaces. Let us consider a four-dimensional scalar field theory in the flat space as an example. The action is given by

$$I = -\int d^4x \left(\frac{1}{2}\eta^{MN}\partial_M\phi(x)\partial_N\phi(x) + V(\phi(x)) \right), \tag{2.1}$$

where $\eta_{MN} = \text{diag}(-1,1,1,1)$. The equation of motion is

$$\eta^{MN}\partial_M\partial_N\phi = V'(\phi). \tag{2.2}$$

The space-like hypersurface which specifies the boundary is parametrized by the $X^M(\sigma^i)$, where $i = 1, 2, 3$. (See Fig.1.) In principle, we can obtain the solution $\bar{\phi}(x)$ to the equation of motion (2.2) which satisfies a boundary condition

$$\bar{\phi}(X^M(\sigma^i)) = \Phi(X^M(\sigma^i)). \tag{2.3}$$

By substituting this solution into I, we obtain the on-shell action in which we are interested,

$$S(X, \Phi(X)) = -\int d^4x \left(\frac{1}{2}\eta^{MN}\partial_M\bar{\phi}(x)\partial_N\bar{\phi}(x) + V(\bar{\phi}(x)) \right). \tag{2.4}$$

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It is in general difficult to solve the equation of motion in the above situation and to obtain the on-shell action directly. Instead, we seek for the Hamilton-Jacobi equations, which are the differential equations satisfied by the on-shell action of this kind.

In order to develop the generalized Hamilton-Jacobi formalism, we consider an action,

$$\tilde{I} = -\int d^4\sigma \det\left(\frac{\partial x(\sigma)}{\partial \sigma}\right)\left(\frac{1}{2}\eta^{MN}\frac{\partial \sigma^\alpha}{\partial x^M(\sigma)}\frac{\partial \sigma^\beta}{\partial x^N(\sigma)}\partial_\alpha \tilde{\phi}(\sigma)\partial_\beta \tilde{\phi}(\sigma) + V(\tilde{\phi}(\sigma))\right), \quad (2.5)$$

where $\sigma^\alpha = (\tau, \sigma^i)$ ($\alpha = 0, 1, 2, 3$, $i = 1, 2, 3$, $\sigma^0 = \tau$). In this action, let both the induced scalar $\tilde{\phi}(\sigma)$ and the coordinates $x^M(\sigma)$ be dynamical variables as in [4]. We regard τ as time in this system. Moreover, let the boundary values of the $x^M(\sigma)$ parametrize the same space-like hypersurface,

$$x^M(T, \sigma^i) = X^M(\sigma^i), \quad (2.6)$$

where T is the boundary value of τ that is the final time. $\tilde{I}$ is invariant under the reparametrization of σ by which both $\tilde{\phi}(\sigma)$ and the $x^M(\sigma)$ are transformed as scalars. If this diffeomorphism is fixed by the gauge fixing condition $x^M(\sigma) = \sigma^M$, $\tilde{I}$ reduces to I. Therefore $\tilde{I}$ and I are equivalent. We will actually see below that the on-shell action of $\tilde{I}$ is equal to that of I.

The equations of motions of $\tilde{I}$ are

$$\partial_\gamma\left[\det\left(\frac{\partial x}{\partial \sigma}\right)\left(\frac{\partial \sigma^\gamma}{\partial x^L}\left(\frac{1}{2}\eta^{MN}\frac{\partial \sigma^\alpha}{\partial x^M}\frac{\partial \sigma^\beta}{\partial x^N}\partial_\alpha \tilde{\phi}\partial_\beta \tilde{\phi} + V(\tilde{\phi})\right) - \frac{\partial \sigma^\alpha}{\partial x^L}\eta^{MN}\frac{\partial \sigma^\gamma}{\partial x^M}\frac{\partial \sigma^\beta}{\partial x^N}\partial_\alpha \tilde{\phi}\partial_\beta \tilde{\phi}\right)\right] = 0,$$

$$\partial_\gamma\left(\det\left(\frac{\partial x}{\partial \sigma}\right)\eta^{MN}\frac{\partial \sigma^\gamma}{\partial x^M}\frac{\partial \sigma^\beta}{\partial x^N}\partial_\beta \tilde{\phi}\right) - \det\left(\frac{\partial x}{\partial \sigma}\right)V'(\tilde{\phi}) = 0. \quad (2.7)$$

Let us consider the classical solution $\bar{x}^M(\sigma)$ and $\bar{\tilde{\phi}}(\sigma)$ which satisfies the boundary condition,

$$\bar{x}^M(T, \sigma^i) = X^M(\sigma^i), \qquad \bar{\tilde{\phi}}(T, \sigma^i) = \tilde{\Phi}(\sigma^i) \equiv \Phi(X(\sigma^i)). \quad (2.8)$$

By substituting this classical solution into $\tilde{I}$, we obtain the on-shell action,

$$\tilde{S}(T, \tilde{\Phi}, X) = -\int_{T_0}^{T} d\tau \int d^3\sigma^i \det\left(\frac{\partial \bar{x}(\sigma)}{\partial \sigma}\right)\left(\frac{1}{2}\eta^{MN}\frac{\partial \sigma^\alpha}{\partial \bar{x}^M(\sigma)}\frac{\partial \sigma^\beta}{\partial \bar{x}^N(\sigma)}\partial_\alpha \bar{\tilde{\phi}}(\sigma)\partial_\beta \bar{\tilde{\phi}}(\sigma) + V(\bar{\tilde{\phi}}(\sigma))\right), \quad (2.9)$$

where T_0 is the initial time. If we define $\bar{\phi}(x)$ by $\bar{\tilde{\phi}}(\sigma) = \bar{\phi}(\bar{x}(\sigma))$ and $x = \bar{x}(\sigma)$, $\bar{\phi}(x)$ satisfies (2.2) and (2.3). Therefore the coordinate transformation $x = \bar{x}(\sigma)$ gives the desired relation

$$S(X, \Phi(X)) = \tilde{S}(T, \tilde{\Phi}, X). \quad (2.10)$$

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Let us derive the Hamilton-Jacobi equations satisfied by $\tilde{S}$. By solving these equations we obtain $\tilde{S}$ and hence S as well. $\tilde{S}$ is a functional of the final time T and the boundary values, $\tilde{\Phi}$ and X. The variation of $\tilde{S}$ with respect to T, $\tilde{\Phi}$, and X is given by

$$
\begin{aligned}
\delta\tilde{S} &= -\int d^3\sigma^i \det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\left(\frac{1}{2}\eta^{MN}\frac{\partial\sigma^\alpha}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\alpha\bar{\tilde{\phi}}\partial_\beta\bar{\tilde{\phi}} + V(\bar{\tilde{\phi}})\right)\delta T\bigg|_{\tau=T} \\
&\quad -\int d^3\sigma^i \det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\eta^{MN}\frac{\partial\tau}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\beta\bar{\tilde{\phi}}\delta\bar{\tilde{\phi}}\bigg|_{\tau=T} \\
&\quad -\int d^3\sigma^i \det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial\bar{x}^L}\left(\frac{1}{2}\eta^{MN}\frac{\partial\sigma^\alpha}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\alpha\bar{\tilde{\phi}}\partial_\beta\bar{\tilde{\phi}} + V(\bar{\tilde{\phi}})\right)\right. \\
&\quad\left. -\frac{\partial\sigma^\alpha}{\partial\bar{x}^L}\eta^{MN}\frac{\partial\tau}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\alpha\bar{\tilde{\phi}}\partial_\beta\bar{\tilde{\phi}}\right)\delta\bar{x}^L\bigg|_{\tau=T}.
\end{aligned}
$$

We have used the equations of motion (2.7) and $\delta\bar{\tilde{\phi}}(T_0,\sigma^i) = \delta\bar{x}(T_0,\sigma^i) = 0$. Moreover, the condition (2.8) implies

$$
\begin{aligned}
\delta\bar{x}^M(T,\sigma^i) &= \delta X^M(\sigma^i) - \partial_\tau\bar{x}^M(T,\sigma^i)\delta T, \\
\delta\bar{\tilde{\phi}}(T,\sigma^i) &= \delta\tilde{\Phi}(\sigma^i) - \partial_\tau\bar{\tilde{\phi}}(T,\sigma^i)\delta T.
\end{aligned} \tag{2.11}
$$

Then,

$$
\begin{aligned}
\frac{\partial\tilde{S}}{\partial T} &= 0, \\
\frac{\delta\tilde{S}}{\delta X^L} &= -\det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial\bar{x}^L}\left(\frac{1}{2}\eta^{MN}\frac{\partial\sigma^\alpha}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\alpha\bar{\tilde{\phi}}\partial_\beta\bar{\tilde{\phi}} + V(\bar{\tilde{\phi}})\right)\right. \\
&\quad\left. -\frac{\partial\sigma^\alpha}{\partial\bar{x}^L}\eta^{MN}\frac{\partial\tau}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\alpha\bar{\tilde{\phi}}\partial_\beta\bar{\tilde{\phi}}\right)\bigg|_{\tau=T}, \\
\frac{\delta\tilde{S}}{\delta\tilde{\Phi}} &= -\det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\eta^{MN}\frac{\partial\tau}{\partial\bar{x}^M}\frac{\partial\sigma^\beta}{\partial\bar{x}^N}\partial_\beta\bar{\tilde{\phi}}\bigg|_{\tau=T}.
\end{aligned} \tag{2.12}
$$

Here we define L_M and γ_{ij} for convenience:

$$
\begin{aligned}
L_M &\equiv \epsilon_{ML_1L_2L_3}\partial_1 X^{L_1}\partial_2 X^{L_2}\partial_3 X^{L_3} = \det\left(\frac{\partial\bar{x}}{\partial\sigma}\right)\frac{\partial\tau}{\partial\bar{x}^M}\bigg|_{\tau=T} \quad (\epsilon_{0123}=1), \\
\gamma_{ij} &\equiv \partial_i X^M\partial_j X^N\eta_{MN}, \qquad \gamma \equiv \det(\gamma_{ij}).
\end{aligned} \tag{2.13}
$$

From (2.12) and (2.13), noting that

$$
\begin{aligned}
\partial_i X^M L_M &= 0, \quad \eta^{MN}L_M L_N = -\gamma, \\
\frac{\partial\sigma^\alpha}{\partial\bar{x}^M}\partial_\alpha\bar{\tilde{\phi}}\bigg|_{\tau=T} &= \frac{1}{\gamma}\frac{\delta\tilde{S}}{\delta\tilde{\Phi}}L_M + \gamma^{ij}\partial_i\tilde{\Phi}\eta_{MN}\partial_j X^N,
\end{aligned} \tag{2.14}
$$

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we obtain the Hamilton-Jacobi equations

$$\frac{\delta \tilde{S}}{\delta X^M} = -\left(\frac{1}{2} \left(\frac{1}{\sqrt{\gamma}} \frac{\delta \tilde{S}}{\delta \tilde{\Phi}} \right)^2 + \frac{1}{2} \gamma^{ij} \partial_i \tilde{\Phi} \partial_j \tilde{\Phi} + V(\tilde{\Phi}) \right) L_M - \gamma^{ij} \partial_i \tilde{\Phi} \eta_{MN} \partial_j X^N \frac{\delta \tilde{S}}{\delta \tilde{\Phi}}, \quad (2.15)$$

$$\frac{\partial \tilde{S}}{\partial T} = 0. \quad (2.16)$$

It is instructive to rederive the above Hamilton-Jacobi equations in the canonical formalism following Dirac [4]. $\tilde{I}$ is rewritten in the canonical formalism as follows:

$$\tilde{I} = \int d\tau d^3\sigma^i \left(P_M \partial_\tau x^M + P_{\tilde{\phi}} \partial_\tau \tilde{\phi} - C^M \Sigma_M(x, \tilde{\phi}, P_L, P_{\tilde{\phi}}) \right), \quad (2.17)$$

where

$$\Sigma_M(x, \tilde{\phi}, P_L, P_{\tilde{\phi}}) = P_M + \left(\frac{1}{2} \left(\frac{1}{\sqrt{\gamma}} P_{\tilde{\phi}} \right)^2 + \frac{1}{2} \gamma^{ij} \partial_i \tilde{\phi} \partial_j \tilde{\phi} + V(\tilde{\phi}) \right) L_M + P_{\tilde{\phi}} \partial_i \tilde{\phi} \gamma^{ij} \eta_{MN} \partial_j x^N, \quad (2.18)$$

and C^M are Lagrange multipliers. The constraints $\Sigma_M = 0$ are first-class ones [1] and come from the invariance under the reparametrization of σ. By applying the relation between the on-shell action and the canonical momenta,

$$P_M = \frac{\delta \tilde{S}}{\delta X^M}, \quad P_{\tilde{\phi}} = \frac{\delta \tilde{S}}{\delta \tilde{\phi}}, \quad (2.19)$$

to this constraint, we obtain (2.15). (See section 2 in [2].) On the other hand, the ordinary Hamilton-Jacobi equation $\frac{\delta \tilde{S}}{\delta T} + H = 0$ reduces to (2.16), since the Hamiltonian $H = \int d^3\sigma^i C^M \Sigma_M$ vanishes on shell.

3   Application to gravitational systems

In this section, we consider the same problem as the previous section in gravitational systems. Let us consider as an example a four-dimensional gravity given by

$$I = \int d^4x \sqrt{-g(x)} \left(R(x) - \frac{1}{2} g^{MN}(x) \partial_M \phi(x) \partial_N \phi(x) + V(\phi(x)) \right). \quad (3.1)$$

As before we consider the classical solution $\bar{g}_{MN}(x)$ and $\bar{\phi}(x)$ which satisfies the boundary condition

$$\frac{\partial X^M(\sigma^k)}{\partial \sigma^i} \frac{\partial X^N(\sigma^k)}{\partial \sigma^j} \bar{g}_{MN}(X(\sigma^k)) = \frac{\partial X^M(\sigma^k)}{\partial \sigma^i} \frac{\partial X^N(\sigma^k)}{\partial \sigma^j} G_{MN}(X(\sigma^k)),$$
$$\bar{\phi}(X(\sigma^k)) = \Phi(X(\sigma^k)), \quad (3.2)$$

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[1] Indeed, the Poisson brackets between the Σ_M are $\{\Sigma_M(\tau, \sigma^i), \Sigma_N(\tau, \sigma'^i)\} = 0$.

where the $X^M(\sigma)$ parametrize an space-like hypersurface.[2] Note that this boundary condition is invariant under general coordinate transformations of x. We also consider the on-shell action

$$S(X, G(X), \Phi(X)) = \int d^4x \sqrt{-\bar{g}(x)} \left(\bar{R}(x) - \frac{1}{2}\bar{g}^{MN}(x)\partial_M\bar{\phi}(x)\partial_N\bar{\phi}(x) + V(\bar{\phi}(x)) \right), \quad (3.3)$$

which is an analogue of (2.4).

Note that we can obtain (2.5) from (2.1) by performing a general coordinate transformation $x = x(\sigma)$. Similarly, we obtain from (3.1)

$$\tilde{I} = \int d^4\sigma \sqrt{-\tilde{g}(\sigma)} \left(\tilde{R}(\sigma) - \frac{1}{2}\tilde{g}^{\alpha\beta}(\sigma)\partial_\alpha\tilde{\phi}(\sigma)\partial_\beta\tilde{\phi}(\sigma) + V(\tilde{\phi}(\sigma)) \right). \quad (3.4)$$

Note that $\tilde{I}$ does not depend on the $x^M(\sigma)$ and takes the same form as I, because I is invariant under the general coordinate transformations. Now we consider the classical solution $\bar{\tilde{g}}_{\alpha\beta}(\sigma)$ and $\bar{\tilde{\phi}}(\sigma)$ which satisfies the boundary condition on the fixed-time hypersurface,

$$\begin{aligned}
\bar{\tilde{g}}_{ij}(T, \sigma^k) &= \tilde{G}_{ij}(\sigma^k) \equiv \frac{\partial \bar{x}^M(T, \sigma^k)}{\partial \sigma^i}\frac{\partial \bar{x}^N(T, \sigma^k)}{\partial \sigma^j}G_{MN}(X(\sigma^k)), \\
\bar{\tilde{\phi}}(T, \sigma^k) &= \tilde{\Phi}(\sigma^k) \equiv \Phi(X(\sigma^k)), \\
\bar{x}^M(T, \sigma^k) &= X^M(\sigma^k),
\end{aligned} \quad (3.5)$$

Then the on-shell action of $\tilde{I}$ is

$$\tilde{S}(T, \tilde{G}, \tilde{\Phi}) = \int_{T_0}^{T} d\tau \int d^3\sigma^i \sqrt{-\bar{\tilde{g}}(\sigma)} \left(\bar{\tilde{R}}(\sigma) - \frac{1}{2}\bar{\tilde{g}}^{\alpha\beta}(\sigma)\partial_\alpha\bar{\tilde{\phi}}(\sigma)\partial_\beta\bar{\tilde{\phi}}(\sigma) + V(\bar{\tilde{\phi}}(\sigma)) \right). \quad (3.6)$$

As in the previous section, if we define $\bar{g}_{MN}(x)$ and $\bar{\phi}(x)$ by $\bar{g}_{MN}(\bar{x}(\sigma)) = \frac{\partial \sigma^\alpha}{\partial \bar{x}^M}\frac{\partial \sigma^\beta}{\partial \bar{x}^N}\bar{\tilde{g}}_{\alpha\beta}(\sigma)$, $\bar{\phi}(\bar{x}(\sigma)) = \bar{\tilde{\phi}}(\sigma)$ and $x = \bar{x}(\sigma)$, $\bar{g}_{MN}(x)$ and $\bar{\phi}(x)$ satisfy the equation of motion of I and the boundary condition (3.2). It follows again that the on-shell actions are equivalent:

$$S(X, G(X), \Phi(X)) = \tilde{S}(T, \tilde{G}, \tilde{\Phi}). \quad (3.7)$$

Contrary to the case of the flat space, the Hamilton-Jacobi equations of $\tilde{I}$ clearly take the same forms as the ordinary ones of I, which are satisfied by the on-shell action with the boundary values of the fields given on the fixed-time hypersurface. Therefore if one knows

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[2]In fact, we must add the Gibbons-Hawking term to the action in order to impose a boundary condition such as (3.2) consistently. However, the following argument is still valid after adding this term.

the on-shell action with the boundary values of the fields given on the fixed-time hypersurface, one can obtain the on-shell action with those given on an arbitrary space-like hypersurface. Note that this consequence directly comes from the facts that I is invariant under general coordinate transformations and that an arbitrary space-like hypersurface can be transformed to a fixed-time hypersurface by a general coordinate transformation.

Finally we apply the above argument to our results in [1, 2]. We reported in [1, 2] that the D-brane and M-brane effective actions are on-shell actions of supergravities, which are defined on fixed-radial-time hypersurfaces. Obviously, the above argument holds even when we replace the time with the radial time that is a space-like direction. It follows that the D-brane and M-brane effective actions given by arbitrary embedding functions are also on-shell actions in supergravities. We take the D3-brane case as an example below and write down the explicit form of the on-shell action. The on-shell actions corresponding to the other branes can be written down explicitly in the same way.

We start with the five-dimensional gravity which we obtained in [1, 2] by reducing type IIB supergravity on S^5,

$$I_5 = \int d^5x \sqrt{-h} \left[e^{-2\phi + \frac{5}{4}\rho} \left(R + 4\partial_\alpha \phi \partial^\alpha \phi + \frac{5}{4}\partial_\alpha \rho \partial^\alpha \rho - 5\partial_\alpha \phi \partial^\alpha \rho - \frac{1}{2}|H_3|^2 \right) \right.$$
$$\left. - \frac{1}{2}e^{\frac{5}{4}\rho} \left(|F_1|^2 + |F_3 + c_0 H_3|^2 + |F_5 + c_2 H_3|^2 \right) + e^{-2\phi + \frac{3}{4}\rho} R^{(S^5)} \right], (3.8)$$

where $H_3 = db_2$, $F_{p+2} = dc_{p+1}$ ($p = -1, 1, 3$). The x^M ($M = 0, \cdots, 4$) are five-dimensional coordinates. h_{MN}, ϕ, ρ, b_2 and c_{p+1} are the five-dimensional metric, the dilaton, the warp factor of S^5, the Kalb-Ramond field and the R-R $(p+1)$-form, respectively. $R^{(S^5)}$ is the constant curvature of S^5.

In this case, we are interested in on-shell actions with the boundary values of fields given on general *time-like* hypersurfaces, since we interpret these hypersurfaces as the worldvolumes of D3-brane. Let $X^M(\sigma^\mu)$ ($\mu = 0, \cdots, 3$) be embedding functions satisfying such a hypersurface. We consider the classical solution $\bar{h}_{MN}(x)$, $\bar{\phi}(x)$, $\bar{\rho}(x)$, $\bar{b}_{MN}(x)$ and $\bar{c}_{M_1 \cdots M_{p+1}}(x)$ which satisfies the boundary condition

$$\frac{\partial X^M(\sigma^\lambda)}{\partial \sigma^\mu} \frac{\partial X^N(\sigma^\lambda)}{\partial \sigma^\nu} \bar{h}_{MN}(X(\sigma^\lambda)) = \frac{\partial X^M(\sigma^\lambda)}{\partial \sigma^\mu} \frac{\partial X^N(\sigma^\lambda)}{\partial \sigma^\nu} G_{MN}(X(\sigma^\lambda)),$$
$$\bar{\phi}(X(\sigma^\mu)) = \Phi(X(\sigma^\mu)),$$
$$\bar{\rho}(X(\sigma^\mu)) = \Upsilon(X(\sigma^\mu)),$$

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$$\frac{\partial X^M(\sigma^\lambda)}{\partial\sigma^\mu}\frac{\partial X^N(\sigma^\lambda)}{\partial\sigma^\nu}\bar{b}_{MN}(X(\sigma^\lambda)) = \frac{\partial X^M(\sigma^\lambda)}{\partial\sigma^\mu}\frac{\partial X^N(\sigma^\lambda)}{\partial\sigma^\nu}B_{MN}(X(\sigma^\lambda)),$$

$$\frac{\partial X^{M_1}(\sigma^\nu)}{\partial\sigma^{\mu_1}}\cdots\frac{\partial X^{M_{p+1}}(\sigma^\nu)}{\partial\sigma^{\mu_{p+1}}}\bar{c}_{M_1\cdots M_{p+1}}(X(\sigma^\nu))$$

$$= \frac{\partial X^{M_1}(\sigma^\nu)}{\partial\sigma^{\mu_1}}\cdots\frac{\partial X^{M_{p+1}}(\sigma^\nu)}{\partial\sigma^{\mu_{p+1}}}C_{M_1\cdots M_{p+1}}(X(\sigma^\nu)). \tag{3.9}$$

As before $\tilde{I}_5$ is obtained by replacing the fields in I_5 with those with tilde, $\tilde{h}_{\alpha\beta}(\sigma^\alpha)$, $\tilde{\phi}(\sigma^\alpha)$, $\tilde{\rho}(\sigma^\alpha)$, $\tilde{b}_2(\sigma^\alpha)$ and $\tilde{c}_{p+1}(\sigma^\alpha)$, where $\sigma^\alpha = (\sigma^\mu, \sigma^4)$ ($\mu = 0, \cdots, 3$). We regard σ^4 as time, and denote the boundary value of σ^4 by U. The classical solution of $\tilde{I}_5$, $\bar{x}^M(\sigma^\alpha)$, $\bar{\tilde{h}}_{\alpha\beta}(\sigma^\gamma)$, $\bar{\tilde{\phi}}(\sigma^\alpha)$, $\bar{\tilde{\rho}}(\sigma^\alpha)$, $\bar{\tilde{b}}_{\alpha\beta}(\sigma^\gamma)$ and $\bar{\tilde{c}}_{\alpha_1\cdots\alpha_{p+1}}(\sigma^\beta)$, corresponding to the above solution of I_5 satisfies the boundary condition

$$\bar{x}^M(\sigma^\mu, U) = X^M(\sigma^\mu),$$

$$\bar{\tilde{h}}_{\mu\nu}(\sigma^\lambda, U) = \tilde{G}_{\mu\nu}(\sigma^\lambda) \equiv \frac{\partial X^M(\sigma^\lambda)}{\partial\sigma^\mu}\frac{\partial X^N(\sigma^\lambda)}{\partial\sigma^\nu}G_{MN}(X(\sigma^\lambda)),$$

$$\bar{\tilde{\phi}}(\sigma^\mu, U) = \tilde{\Phi}(\sigma^\mu) \equiv \Phi(X(\sigma^\mu)),$$

$$\bar{\tilde{\rho}}(\sigma^\mu, U) = \tilde{\Upsilon}(\sigma^\mu) \equiv \Upsilon(X(\sigma^\mu)),$$

$$\bar{\tilde{b}}_{\mu\nu}(\sigma^\lambda, U) = \tilde{B}_{\mu\nu}(\sigma^\lambda) \equiv \frac{\partial X^M(\sigma^\lambda)}{\partial\sigma^\mu}\frac{\partial X^N(\sigma^\lambda)}{\partial\sigma^\nu}B_{MN}(X(\sigma^\lambda)),$$

$$\bar{\tilde{c}}_{\mu_1\cdots\mu_{p+1}}(\sigma^\nu, U) = \tilde{C}_{\mu_1\cdots\mu_{p+1}}(\sigma^\nu) \equiv \frac{\partial X^{M_1}(\sigma^\nu)}{\partial\sigma^{\mu_1}}\cdots\frac{\partial X^{M_{p+1}}(\sigma^\nu)}{\partial\sigma^{\mu_{p+1}}}C_{M_1\cdots M_{p+1}}(X(\sigma^\nu)). \tag{3.10}$$

In order to derive the H-J equations, we perform the ADM decomposition as follows:

$$\begin{aligned}
ds_5^2 &= h_{\alpha\beta}d\sigma^\alpha d\sigma^\beta \\
&= (n^2 + n^\mu n_\mu)(d\sigma^4)^2 + 2n_\mu d\sigma^\mu d\sigma^4 + h_{\mu\nu}d\sigma^\mu d\sigma^\nu, \tag{3.11}
\end{aligned}$$

where n and n_μ are the lapse function and the shift functions, respectively. By adding the Gibbons-Hawking term, $\tilde{I}_5$ can be rewritten in the canonical form as

$$\tilde{I}_5 = \int d^5\sigma\sqrt{-h}(\pi^{\mu\nu}\partial_{\sigma^4}\tilde{h}_{\mu\nu} + \pi_{\tilde{\phi}}\partial_{\sigma^4}\tilde{\phi} + \pi_{\tilde{\rho}}\partial_{\sigma^4}\tilde{\rho} + \pi_{\tilde{b}_2}^{\mu\nu}\partial_{\sigma^4}\tilde{b}_{\mu\nu} + \sum_p \pi_{\tilde{c}_{p+1}}^{\mu_1\cdots\mu_{p+1}}\partial_{\sigma^4}\tilde{c}_{\mu_1\cdots\mu_{p+1}}$$

$$-nH - n_\mu H^\mu - \tilde{b}_{4\mu}Z_{\tilde{b}_2}^\mu - \tilde{c}_{4\mu}Z_{\tilde{c}_2}^\mu - \tilde{c}_{4\mu\nu\lambda}Z_{\tilde{c}_4}^{\mu\nu\lambda}), \tag{3.12}$$

with

$$H = -e^{2\tilde{\phi}-\frac{5}{4}\tilde{\rho}}\left((\pi^{\mu\nu})^2 + \frac{1}{2}\pi_{\tilde{\phi}}^2 + \frac{1}{2}\pi^\mu{}_\mu\pi_{\tilde{\phi}} + \frac{4}{5}\pi_{\tilde{\rho}}^2 + \pi_{\tilde{\phi}}\pi_{\tilde{\rho}} + \left(\pi_{\tilde{b}_2}^{\mu\nu} - \tilde{c}\pi_{\tilde{c}_2}^{\mu\nu} - 6\tilde{c}_{\lambda\rho}\pi_{\tilde{c}_4}^{\mu\nu\lambda\rho}\right)^2\right)$$

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$$-e^{-\frac{5}{4}\tilde{\rho}}\left(\frac{1}{2}\pi_{\tilde{c}_0}{}^2 + (\pi_{\tilde{c}_2}^{\mu\nu})^2 + 12(\pi_{\tilde{c}_4}^{\mu\nu\lambda\rho})^2\right) - \mathcal{L},\tag{3.13}$$

$$H^\mu = -2\nabla_\nu \pi^{\mu\nu} + \pi_{\tilde{\phi}}\partial^\mu\tilde{\phi} + \pi_{\tilde{\rho}}\partial^\mu\tilde{\rho} + \pi_{\tilde{b}_2\nu\lambda}\tilde{H}^{\mu\nu\lambda}$$

$$+\pi_{\tilde{c}_0}\partial^\mu\tilde{c} + \pi_{\tilde{c}_2\nu\lambda}\tilde{F}^{\mu\nu\lambda} + \pi_{\tilde{c}_4\nu\lambda\rho\sigma}(\tilde{F}^{\mu\nu\lambda\rho\sigma} + 4\tilde{c}^{\mu\nu}\tilde{H}^{\lambda\rho\sigma}),\tag{3.14}$$

$$Z_{\tilde{b}_2}^\mu = 2\nabla_\nu \pi_{\tilde{b}_2}^{\mu\nu},\tag{3.15}$$

$$Z_{\tilde{c}_2}^\mu = 2\nabla_\nu \pi_{\tilde{c}_2}^{\mu\nu} - 4\pi_{\tilde{c}_4}^{\mu\nu\lambda\rho}\tilde{H}_{\nu\lambda\rho},\tag{3.16}$$

$$Z_{\tilde{c}_4}^{\mu\nu\lambda} = 4\nabla_\rho \pi_{\tilde{c}_4}^{\mu\nu\lambda\rho},\tag{3.17}$$

where

$$h = \det h_{\mu\nu},$$

$$\tilde{H}_{\mu\nu\lambda} = 3\partial_{[\mu}\tilde{b}_{\nu\lambda]}, \quad \tilde{F}_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\tilde{c}_{\mu_2\cdots\mu_{p+2}]},$$

$$\mathcal{L} = e^{-2\tilde{\phi}+\frac{5}{4}\tilde{\rho}}\left(R^{(4)} + 4\nabla_\mu\nabla^\mu\tilde{\phi} - \frac{5}{2}\nabla_\mu\nabla^\mu\tilde{\rho} - 4\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{15}{8}\partial_\mu\tilde{\rho}\partial^\mu\tilde{\rho} + 5\partial_\mu\tilde{\phi}\partial^\mu\tilde{\rho} - \frac{1}{12}\tilde{H}_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda}\right)$$

$$+e^{\frac{5}{4}\tilde{\rho}}\left(-\frac{1}{2}\partial_\mu\tilde{c}\partial^\mu\tilde{c} - \frac{1}{12}(\tilde{F}_{\mu\nu\lambda} + \tilde{c}\tilde{H}_{\mu\nu\lambda})(\tilde{F}^{\mu\nu\lambda} + \tilde{c}\tilde{H}^{\mu\nu\lambda})\right) + e^{-2\tilde{\phi}+\frac{3}{4}\tilde{\rho}}R^{(S^5)}.\tag{3.18}$$

The H-J equations of this system are

$$\frac{\partial\tilde{S}}{\partial U} = 0,\tag{3.19}$$

and

$$H = 0, \quad H^\mu = 0, \quad Z_{\tilde{b}_2} = 0, \quad Z_{\tilde{c}_2} = 0, \quad Z_{\tilde{c}_4} = 0\tag{3.20}$$

with the following replacements:

$$\tilde{h}_{\mu\nu} \to \tilde{G}_{\mu\nu}, \quad \tilde{\phi} \to \tilde{\Phi}, \quad \tilde{\rho} \to \tilde{\Upsilon}, \quad \tilde{b}_{\mu\nu} \to \tilde{B}_{\mu\nu}, \quad \tilde{c}_{\mu_1\cdots\mu_{p+1}} \to \tilde{C}_{\mu_1\cdots\mu_{p+1}},$$

$$\pi^{\mu\nu} \to \frac{1}{\sqrt{-\tilde{G}}}\frac{\delta\tilde{S}}{\delta\tilde{G}_{\mu\nu}}, \quad \pi_{\tilde{\phi}} \to \frac{1}{\sqrt{-\tilde{G}}}\frac{\delta\tilde{S}}{\delta\tilde{\Phi}}, \quad \pi_{\tilde{\rho}} \to \frac{1}{\sqrt{-\tilde{G}}}\frac{\delta\tilde{S}}{\delta\tilde{\Upsilon}},$$

$$\pi_{\tilde{b}_2}^{\mu\nu} \to \frac{1}{\sqrt{-\tilde{G}}}\frac{\delta\tilde{S}}{\delta\tilde{B}_{\mu\nu}}, \quad \pi_{\tilde{c}_{p+1}}^{\mu_1\cdots\mu_{p+1}} \to \frac{1}{\sqrt{-\tilde{G}}}\frac{\delta\tilde{S}}{\delta\tilde{c}_{\mu_1\cdots\mu_{p+1}}}.\tag{3.21}$$

The H-J equation (3.19) implies that $\tilde{S}$ does not depend on the final time explicitly while the last four equations in (3.20) imply that $\tilde{S}$ is invariant under the diffeomorphism in four dimensions and the $U(1)$ gauge transformations for $\tilde{B}_{\mu\nu}$ and $\tilde{C}_{\mu_1\cdots\mu_{p+1}}$. The first equation $H = 0$ in (3.20) is only a nontrivial one that can be determine the form of $\tilde{S}$.

We solve the H-J equations under the condition that the fields in $\tilde{I}_5$ depend only on σ^4, and denote a solution to the H-J equations under this condition by $\tilde{S}_0$. Then the H-J equation $H = 0$ reduces to

$$
\begin{aligned}
-e^{2\tilde{\Phi}-\frac{5}{4}\tilde{\Upsilon}} & \left(\left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{G}_{\mu\nu}} \right)^2 + \frac{1}{2}\tilde{G}_{\mu\nu} \frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{G}_{\mu\nu}} \frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{\Phi}} + \frac{1}{2} \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{\Phi}} \right)^2 \right. \\
& + \frac{4}{5} \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{\Upsilon}} \right)^2 + \frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{\Phi}} \frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{\Upsilon}} \\
& \left. + \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{B}_{\mu\nu}} - \tilde{C}\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{C}_{\mu\nu}} - 6\tilde{C}_{\lambda\rho}\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{C}_{\mu\nu\lambda\rho}} \right)^2 \right) \\
-e^{-\frac{5}{4}\tilde{\Upsilon}} & \left(\frac{1}{2} \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{C}} \right)^2 + \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{C}_{\mu\nu}} \right)^2 + 12 \left(\frac{1}{\sqrt{-\tilde{G}}} \frac{\delta\tilde{S}_0}{\delta\tilde{C}_{\mu\nu\lambda\rho}} \right)^2 \right) \\
= & \ e^{-2\tilde{\Phi}+\frac{3}{4}\tilde{\Upsilon}} R^{(S^5)}.
\end{aligned} \tag{3.22}
$$

We showed in [1, 2] that the following form is one of the solutions to the reduced H-J equations.

$$
\tilde{S}_0 = \tilde{S}_c + \tilde{S}_{BI} + \tilde{S}_{WZ} \tag{3.23}
$$

with

$$
\begin{aligned}
\tilde{S}_c &= \pm\sqrt{5R^{(S^5)}} \int d^4\sigma\sqrt{-\tilde{G}}e^{-2\tilde{\Phi}+\tilde{\Upsilon}}, \\
\tilde{S}_{BI} &= \beta \int d^4\sigma e^{-\tilde{\phi}}\sqrt{-\det(\tilde{G}_{\mu\nu} + \tilde{\mathcal{F}}_{\mu\nu})}, \\
\tilde{S}_{WZ} &= \pm\beta \left(\int \tilde{C}_4 + \int \tilde{C}_2 \wedge \tilde{\mathcal{F}} + \frac{1}{2}\int \tilde{C}_0\tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}} \right),
\end{aligned} \tag{3.24}
$$

where $\tilde{\mathcal{F}}_{\mu\nu} = \tilde{B}_{\mu\nu} + F_{\mu\nu}$, both $F_{\mu\nu}$ and β are arbitrary constants and the signs in $\tilde{S}_c$ and $\tilde{S}_{WZ}$ can take all combinations. Thus $\tilde{S}_0$ is an on-shell action of $\tilde{I}_5$. The argument in the first half of this section shows that one obtains the corresponding on-shell action S_0 of I_5 by rereading the fields with tilde in (3.23) using (3.10). Hence we conclude that the effective actions of Dp-brane and Mp-brane whose worldvolumes are defined by $p + 1$ embedding functions are on-shell actions of supergravities reduced in $p + 2$ dimensions. This is an important generalization of our results in [1, 2] although its derivation is rather simple.

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4 Discussion

As we have seen in the example of the D3-brane case in the previous section, all we can do at present is to reduce ten-dimensional supergravities or eleven-dimensional supergravity to $(p+2)$-dimensional gravity and obtain the p-brane effective action, whose $(p+1)$-dimensional worldvolume is given by arbitrary embedding functions, as an on-shell action of the $(p+2)$-dimensional gravity. In other words, we can only consider hypersurfaces whose codimension is one. Thus our results in this note are not completely satisfactory, since in string theories one can consider D-branes whose codimension is larger than one. Hence, an issue we should next study is an 'on-shell action' that one obtains when one takes a hypersurface whose codimension is larger than one, specifies the 'boundary' values on the hypersurface and substitutes into the action the classical solution satisfying the 'boundary' condition. We need to develop a formalism that gives such an 'on-shell action' and see whether the p-brane effective action is an 'on-shell action' of a $(p+k)$-dimensional gravity which is obtained by reducing ten-dimensional supergravities or eleven-dimensional supergravity, where $2 < k \leq 10 - p$ (or $11 - p$).[3]

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[3]In [5], the Nambu-Goto action and its corrections were derived from four-dimensional field theories. Perhaps these works will give a hint to the above issue.

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